

# THE HEAT EQUATION UNDER CONDITIONS ON THE MOMENTS IN HIGHER DIMENSIONS

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**ABSTRACT.** We study a realization of the Laplacian on an  $n$ -dimensional cube on which we impose different classes of integral conditions, instead of usual boundary ones. Well-posedness results for the heat equation under conditions on the moments of order 0 and 1 had been known so far only in the 1-dimensional case. We show that in the  $n$ -dimensional situation the heat equation with such integral conditions is still well-posed, if we suitably relax the notion of solution. This is done by showing that both said realizations of the Laplacian generate an analytic semigroup on a suitable space of distributions.

## 1. INTRODUCTION

Fifty years ago, J.R. Cannon has suggested in [3] that in considerations of certain mechanical models it is reasonable to study diffusion equations dropping a boundary condition and replacing it by a condition on the moment of order 0 (i.e., on the mass) of the unknown. Cannon's analysis was limited to the one-dimensional case and, while his setting has been significantly generalized over the years (cf. [9, § 1] for a historical overview), to the best of our knowledge only a few tentative extensions of the above conditions for heat or wave equations on higher dimensional domains have been proposed in the literature. In fact we are only aware of the preliminary results in [8, 10]. The main problem is obviously that in case of a bounded domain in dimension  $N \geq 2$  (unlike in the 1-dimensional case) infinitely many (boundary) conditions are necessary to determine a solution, while conditions on the moments of order 0 and 1 yield only two conditions.

In this note, we solve this long-standing problem by introducing a general setting which turns out to yield the proper extensions of the 1-dimensional moment conditions studied in [2, 9]. This allows to define two diffusion-type operators  $A$  and  $\tilde{A}$  and to study the associated parabolic problem. We stress that neither of these operators is a classical Laplacian: Rather, they are isomorphic images of the Laplacian with values in a suitable space of distributions of  $H^{-1}$ -type. An analogous problem has been already observed in [9]. Roughly speaking,  $A$  will turn out to be an operator with conditions on the moments of order 0 and 1, whereas  $\tilde{A}$  will be an operator with periodic boundary conditions and a condition on the moment of order 0 only. This is discussed in detail in Section 4.

Even if  $A$  and  $\tilde{A}$  only agree with the Laplacian in the sense of distributions, in the special case of  $N = 1$  it has been shown in [9, §§ 3–4] that the solution of the abstract Cauchy problems associated with them also solve the classical heat equation. This is done by determining the domain of some powers of this operator; showing that their elements are smooth enough to ensure that  $A$  acts on them as the usual second derivative; and finally recalling that an analytic semigroups has a smoothing effect, ensuring that all initial data are mapped into solutions of the standard heat equation (with integral conditions).

It seems that a similar strategy is not successful in the present higher dimensional context. In fact, a direct computation show that in general the elements of  $D(A^2)$  or  $D(\tilde{A}^2)$  are not any better than  $L^2$ . Nevertheless, we will eventually show in Section 5 that the semigroup generated by  $A$  actually solves the usual heat equation (with moment conditions) – at least in a suitably weak, distributional sense.

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## 2. TWO GEL'FAND TRIPLES AND A RELATED ISOMORPHISM

The space  $H^1(T^N)$  is defined as usual, where we denote by  $T^N$  the  $N$ -dimensional torus, the space  $H^{-1}(T^N)$  is its dual with respect to the pivot space  $L^2(T^N) = L^2((0, 1)^N)$ . Similarly  $H^{-1}((0, 1)^N) = (H_0^1(0, 1)^N)'$  also with respect to the pivot space  $L^2(T^N)$ . Throughout this article we will denote

$$\mu_0(h) := \langle h, 1 \rangle, \quad h \in H^{-1}(T^N),$$

the mean – i.e., the moment of order 0 – of a distribution  $h$ .

For further uses, we need to show that  $H^{-1}((0, 1)^N) := (H_0^1((0, 1)^N))'$  is isomorphic to the annihilator

$$H := \{w \in H^{-1}(T^N) : \langle w, v \rangle = 0, \quad \forall v \in H_0\},$$

of

$$H_0 := \{v \in H^1(T^N) : \Delta v = 0 \text{ in } \mathcal{D}'((0, 1)^N)\}.$$

For the sake of later reference, let us also introduce the space

$$H_1 := \{v \in H^1((0, 1)^N) : \Delta v = 0 \text{ in } \mathcal{D}'((0, 1)^N)\}.$$

**Remark 2.1.** In other words,  $H_0$  is the space of periodic functions that are weakly harmonic, whereas  $H_1$  is the space of weakly harmonic function on the unit cube. It is by definition clear that  $H_0$  is a subspace of  $H_1$ . In fact, it will in general be a proper subspace: For  $N = 1$  we e.g. find that  $H_0$  and  $H_1$  are isomorphic to the spaces  $\mathbb{P}_0$  and  $\mathbb{P}_1$  of polynomials of degree 0 and of degree less than or equal to 1, respectively.

For this purpose we define a mapping

$$\text{Id} : H^{-1}(T^N) \ni w \mapsto \text{Id } w \in H^{-1}((0, 1)^N),$$

by

$$\langle \text{Id } w, v \rangle_{H^{-1}((0, 1)^N) - H_0^1((0, 1)^N)} := \langle w, v \rangle_{H^{-1}(T^N) - H^1(T^N)}, \quad \forall v \in H_0^1((0, 1)^N).$$

Denote by  $\text{Id}_m$  the restriction of  $\text{Id}$  to the space  $H$ , namely

$$\text{Id}_m : H \ni w \mapsto \text{Id } w \in H^{-1}((0, 1)^N).$$

While  $H^{-1}(T^N)$  is not a subspace of  $H^{-1}((0, 1)^N)$ , and in fact the identification operator  $\text{Id}$  cannot be seen as the canonical injection. In order to explain this, we need some preparation.

Denote by

$$H^{\frac{1}{2}}(\partial T^N) := \{\gamma_0 v : v \in H^1(T^N)\},$$

where  $\gamma_0$  is the trace operator from  $H^1((0, 1)^N)$  into  $H^{\frac{1}{2}}(\partial(0, 1)^N)$ . The space  $H^{\frac{1}{2}}(\partial T^N)$  is smaller than  $H^{\frac{1}{2}}(\partial(0, 1)^N)$  due to the periodicity assumption on elements from  $H^1(T^N)$ . Nevertheless  $H^{\frac{1}{2}}(\partial T^N)$  is a Hilbert space with the induced norm

$$\|\varphi\|_{H^{\frac{1}{2}}(\partial T^N)} := \inf_{\substack{v \in H^1(T^N) \\ \varphi = \gamma_0 v}} \|v\|_{H^1(T^N)}.$$

**Lemma 2.2.** *For all  $\varphi \in H^{\frac{1}{2}}(\partial T^N)$ , there exists a unique  $R\varphi \in H_0$  such that*

$$\gamma_0 R\varphi = \varphi \text{ on } \partial(0, 1)^N.$$

*In other words,  $H_0$  is isomorphic to  $H^{\frac{1}{2}}(\partial T^N)$ .*

The existence of the right inverse of the trace operator, seen as an operator from the space of weakly harmonic functions to the space of  $H^{\frac{1}{2}}$ -functions over the boundary of a domain, is more or less classical - see e.g. the much more general discussion in [7, § 2.7] or the alternative approach in [5, Lemma 1.2]. Here we provide a direct and elementary proof.

*Proof.* As the mapping

$$H^1(T^N) \ni v \mapsto \gamma_0 v \in H^{\frac{1}{2}}(\partial T^N)$$

is continuous, linear and surjective (e.g. by an application of [1, Thm. 7.39] for  $p = 2$ ), it admits a continuous right inverse  $R_1$ , namely there exists a constant  $C > 0$  such that for any  $\varphi \in H^{\frac{1}{2}}(\partial T^N)$ , there exists a unique  $R_1\varphi \in H^1(T^N)$  such that

$$\gamma_0 R_1\varphi = \varphi \text{ on } \partial(0, 1)^N,$$

and

$$\|R_1\varphi\|_{H^1(T^N)} \leq C\|\varphi\|_{H^{\frac{1}{2}}(\partial T^N)}.$$

Now for an arbitrary  $\varphi \in H^{\frac{1}{2}}(\partial T^N)$ , consider the unique solution  $w_0 \in H_0^1((0,1)^N)$  of

$$\int_{(0,1)^N} \nabla w_0 \cdot \nabla w \, dx = - \int_{(0,1)^N} \nabla(R_1\varphi) \cdot \nabla w \, dx, \quad \forall w \in H_0^1((0,1)^N),$$

which can be found applying the Theorem of Riesz–Fréchet. The conclusion finally follows by defining

$$R\varphi = R_1\varphi + w_0.$$

This concludes the proof.  $\square$

In the following, we denote by  $\text{supp } w$  the support of a distribution  $w \in \mathcal{D}'(T^N)$ .

**Lemma 2.3.** *The linear and continuous mapping  $\text{Id}$  is not injective, since*

$$\ker \text{Id} = \{w \in H^{-1}(T^N) : \text{supp } w \subset \partial(0,1)^N\}.$$

*However,  $\text{Id}_m$  is an isomorphism.*

*Proof.* One sees that  $w \in \ker \text{Id}$  if and only if

$$\langle w, v_1 \rangle_{H^{-1}(T^N)-H^1(T^N)} = 0, \quad \forall v_1 \in H_0^1((0,1)^N).$$

Hence the support of  $w$  is indeed included into  $\partial(0,1)^N$ . But in fact we have

$$(2.1) \quad v_1 := v - R(\gamma_0 v) \in H_0^1((0,1)^N), \quad \forall v \in H^1(T^N).$$

Hence if  $w \in \ker \text{Id}$ , then

$$\begin{aligned} \langle w, v \rangle_{H^{-1}(T^N)-H^1(T^N)} &= \langle w, v_1 \rangle_{H^{-1}(T^N)-H^1(T^N)} + \langle w, R(\gamma_0 v) \rangle_{H^{-1}(T^N)-H^1(T^N)} \\ &= \langle w, R(\gamma_0 v) \rangle_{H^{-1}(T^N)-H^1(T^N)}. \end{aligned}$$

Let us now prove that  $\text{Id}_m$  is an isomorphism. It follows from the last identity that  $\text{Id}_m$  is injective, since in particular for  $w \in \ker \text{Id}_m$  and  $v$  in  $H^1(T^N)$  we obtain

$$\langle w, v \rangle_{H^{-1}(T^N)-H^1(T^N)} = \langle w, R(\gamma_0 v) \rangle_{H^{-1}(T^N)-H^1(T^N)} = 0,$$

since  $R(\gamma_0 v) \in H_0$ .

But it is also surjective because given  $w_1 \in H^{-1}((0,1)^N)$ , we can define  $w \in H^{-1}(T^N)$  by

$$\langle w, v \rangle_{H^{-1}(T^N)-H^1(T^N)} = \langle w_1, v - R(\gamma_0 v) \rangle_{H^{-1}((0,1)^N)-H_0^1((0,1)^N)}, \quad \forall v \in H^1(T^N).$$

Now,  $w \in H$  because Lemma 2.2 yields  $v = R(\gamma_0 v)$  for  $v \in H_0$ , and since

$$\text{Id}_m w = w_1$$

the surjectivity follows.  $\square$

The previous lemma shows that

$$\text{Id}_m^{-1} = (Id - R\gamma_0)^*.$$

Define

$$\begin{aligned} \tilde{V} &:= \{f \in L^2((0,1)^N) : (f|g) = 0, \quad \forall g \in H_0\}, \\ V &:= \{f \in L^2((0,1)^N) : (f|g) = 0, \quad \forall g \in H_1\}, \end{aligned}$$

that are clearly two closed subspaces of  $L^2((0,1)^N)$ . Indeed by introducing  $V_0$  (resp.  $V_1$ ) as the closure of  $H_0$  (resp.  $H_1$ ) in  $L^2((0,1)^N)$ , one sees that  $\tilde{V} = V_0^\perp$  (resp.  $V = V_1^\perp$ ). That is,  $\tilde{V}$  is the orthogonal complement of  $V_0$  in  $L^2((0,1)^N)$ , while  $V$  is the orthogonal complement of  $V_1$ . As  $H_0 \subset H_1$  and hence  $V_0 \subset V_1$ , we conclude that  $V \subset \tilde{V}$ .

Below we need a certain characterization of  $V_0$ . For that purpose, we can notice that  $V_0$  is trivially included into the domain of  $\Delta_{L^2}$  – the part of  $\Delta$  in  $L^2((0,1)^N)$  –, i.e., into

$$D(\Delta_{L^2}) := \{v \in L^2((0,1)^N) : \Delta v \in L^2((0,1)^N)\},$$

that is a Hilbert space equipped with the natural graph norm

$$\left( \|v\|_{L^2((0,1)^N)}^2 + \|\Delta v\|_{L^2((0,1)^N)}^2 \right)^{\frac{1}{2}}.$$

It follows from the general theory of interior elliptic regularity that if  $u \in D(\Delta_{L^2})$ , then in general one only has  $u \in H_{\text{loc}}^2((0,1)^N)$ , hence a trace of  $u$  need not exist as an element of  $L^2(\partial(0,1)^N)$ . However, we can give a meaning of its trace on each faces but in a space of distributions. Indeed following [6, Thm 1.5.3.4], the space  $\mathcal{D}([0,1]^N)$  – the set of the restriction of elements of  $\mathcal{D}(\mathbb{R}^N)$  to  $(0,1)^N$  – is dense in  $D(\Delta_{L^2})$  and for all  $i = 1, \dots, n$ , the trace operator

$$\gamma_{i\pm} : v \mapsto v|_{\Gamma_{i\pm}},$$

which is certainly defined for  $v \in \mathcal{D}([0,1]^N)$ , has a unique continuous extension from  $D(\Delta_{L^2})$  into  $(\tilde{H}^{\frac{1}{2}}(\Gamma_{i\pm}))'$ . We denote also this extension by  $\gamma_{i\pm}$ . Here and below we are denoting by  $\Gamma_{i\pm}$  the faces of the hypercube, which are defined by

$$\begin{aligned} \Gamma_{i-} &:= \{x \in [0,1]^N : x_i = 0 \text{ and } x_j \in (0,1), \forall j \neq i\}, \\ \Gamma_{i+} &:= \{x \in [0,1]^N : x_i = 1 \text{ and } x_j \in (0,1), \forall j \neq i\}. \end{aligned}$$

Furthermore  $\tilde{H}^{\frac{1}{2}}(\Gamma_{i\pm})$  denotes the subspace of elements  $w \in H^{\frac{1}{2}}(\Gamma_{i\pm})$  such that  $\tilde{w}$ , its extension by zero outside  $\Gamma_{i\pm}$ , belongs to  $H^{\frac{1}{2}}(\partial(0,1)^N)$ .

**Definition 2.4.** Let  $s > \frac{1}{2}$ . We call a function  $v \in D(\Delta_{L^2}) \cup H^s((0,1)^N)$  periodic if it satisfies

$$(2.2) \quad \gamma_{i-}v = T_i \gamma_{i+}v \text{ in } \left( \tilde{H}^{\frac{1}{2}}(\Gamma_{i+}) \right)', \quad \forall i = 1, \dots, n.$$

Here  $T_i : \left( \tilde{H}^{\frac{1}{2}}(\Gamma_{i+}) \right)' \rightarrow \left( \tilde{H}^{\frac{1}{2}}(\Gamma_{i-}) \right)'$  is the operator defined by means of

$$\langle T_i \psi, \varphi \rangle := \langle \psi, T_i^* \varphi \rangle, \quad \forall \psi \in \left( \tilde{H}^{\frac{1}{2}}(\Gamma_{i+}) \right)', \varphi \in \tilde{H}^{\frac{1}{2}}(\Gamma_{i-}),$$

with

$$(T_i^* \varphi)(x_1, \dots, 1, \dots, x_n) := \varphi(x_1, \dots, 0, \dots, x_n) \quad \text{for a.e. } (x_1, \dots, 1, \dots, x_n) \in \Gamma_{i+}.$$

**Lemma 2.5.** One has

$$V_0 \subset \{v \in L^2((0,1)^N) : \Delta v = 0 \text{ in } \mathcal{D}'((0,1)^N) \text{ satisfying (2.2)}\}.$$

*Proof.* As  $H_0 \subset H^1(T^N)$ , any  $u \in H_0$  clearly satisfies (2.2). Since there exists  $C > 0$  such that

$$\|\gamma_{i-}v - T_i \gamma_{i+}v\| \leq C \left( \|v\|_{L^2((0,1)^N)}^2 + \|\Delta v\|_{L^2((0,1)^N)}^2 \right)^{\frac{1}{2}}, \quad \forall v \in D(\Delta_{L^2}),$$

we directly conclude that any  $v \in V_0$  still satisfies (2.2).  $\square$

**Remark 2.6.** Note that in particular any function in  $H^1(T^N)$  is periodic in the sense of the above definition, and by Lemma 2.5 so is any element of  $H^1(T^N) + V_0$ , too.

### 3. AN EQUIVALENT INNER PRODUCT IN $H^{-1}(T^N)$

**Lemma 3.1.** For all  $f \in H^{-1}(T^N)$  there exists a unique  $u_f \in H_m^1(T^N)$  such that

$$\text{div} \nabla u_f = f - \mu_0(f).$$

An equivalent inner product in  $H^{-1}(T^N)$  is therefore given by

$$(\nabla u_f | \nabla u_g)_{L^2(T^N)^N} + \mu_0(f) \mu_0(\bar{g}), \quad f, g \in H^{-1}(T^N).$$

In the proof we will need a closed subspace  $H_m^1(T^N)$  of  $H^1(T^N)$  defined by

$$H_m^1(T^N) := \left\{ u \in H^1(T^N) : \int_{(0,1)^N} u(x) dx = 0 \right\}.$$

By the Poincaré inequality (see e.g. [4, Thm. 5.8.1]) we know that

$$u \mapsto \|\nabla u\|_{L^2(T^N)^N}$$

defines a norm on  $H_m^1(T^N)$  that is equivalent to the standard  $H^1$ -norm.

*Proof.* Given  $f \in H^{-1}(T^N)$ , we set

$$f_0 = f - \langle f, 1 \rangle,$$

where here and below the duality bracket is between  $H^{-1}(T^N)$  and  $H^1(T^N)$ . Hence  $f_0 \in H^{-1}(T^N)$  and satisfies

$$\langle f_0, 1 \rangle = 0.$$

Now by the Theorem of Riesz–Fréchet there exists a unique solution  $u_f \in H_m^1(T^N)$  of

$$(3.1) \quad \int_{(0,1)^N} \nabla u_f \cdot \nabla \bar{v} dx = -\langle f_0, v \rangle, \quad \forall v \in H_m^1(T^N).$$

Since  $\langle f_0, 1 \rangle = 0$ , this identity remains valid on the whole  $H^1(T^N)$ , namely

$$(3.2) \quad \int_{(0,1)^N} \nabla u_f \cdot \nabla \bar{v} dx = -\langle f_0, v \rangle, \quad \forall v \in H^1(T^N).$$

By choosing smooth enough test functions  $v$ , we see that

$$\operatorname{div} \nabla u_f = f_0 \text{ in } \mathcal{D}'((0,1)^N),$$

or equivalently

$$(3.3) \quad f = \operatorname{div} \nabla u_f + \langle f, 1 \rangle.$$

According to (3.2), we have

$$\|\nabla u_f\|_{L^2(T^N)^N}^2 = -\langle f_0, u_f \rangle \leq \|f_0\|_{H^{-1}(T^N)} \|u_f\|_{H^1(T^N)},$$

and by the equivalence of norm mentioned before, we get

$$\|\nabla u_f\|_{L^2(T^N)^N} \lesssim \|f_0\|_{H^{-1}(T^N)} \lesssim \|f\|_{H^{-1}(T^N)}.$$

Conversely (3.2) is equivalent to

$$\langle f, v \rangle = - \int_{(0,1)^N} \nabla u_f \cdot \nabla \bar{v} dx + \langle f, 1 \rangle \overline{\langle v, 1 \rangle}, \quad \forall v \in H^1(T^N).$$

Consequently

$$\begin{aligned} \|f\|_{H^{-1}(T^N)} &= \sup_{\|v\|_{H^1(T^N)}=1} |\langle f, v \rangle| \\ &\leq \sup_{\|v\|_{H^1(T^N)}=1} \left( \left| \int_{(0,1)^N} \nabla u_f \cdot \nabla \bar{v} dx \right| + |\langle f, 1 \rangle| |\langle v, 1 \rangle| \right) \\ &\lesssim \|\nabla u_f\|_{L^2(T^N)^N} + |\langle f, 1 \rangle|. \end{aligned}$$

This completely proves the assertion.  $\square$

Let us briefly compare the theory we have just developed with that introduced in [9] in the one-dimensional setting. We define the “primitive”  $P_N f \in L^2(T^N)^N$  of any  $f \in H^{-1}(T^N)$  by

$$P_N f := \nabla u_f + \frac{\mu_0(f)}{N} \left( \vec{x} - \frac{\vec{1}}{2} \right), \quad \forall f \in H^{-1}(T^N)$$

where  $\vec{x}$  and  $\frac{\vec{1}}{2}$  denote the vector-valued functions

$$\vec{x} : (x_1, \dots, x_N) \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \quad \text{and} \quad \frac{\vec{1}}{2} : (x_1, \dots, x_N) \mapsto \frac{1}{2} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

respectively. From this expression, we see that

$$\operatorname{div} P_N f = f \text{ in } \mathcal{D}'((0,1)^N),$$

and

$$\langle P_N f, \alpha \rangle = 0, \quad \forall \alpha \in \mathbb{C}^N.$$

It finally follows from Lemma 3.1 that the inner product

$$(f|g)_{H^{-1}(T^N)} = (P_N f | P_N g)_{L^2(T^N)} + \mu_0(f)\mu_0(\bar{g}),$$

induces a norm equivalent to the standard norm of  $H^{-1}(T^N)$ .

Note further that if  $f \in L^2(T^N)$ , then the solution  $u_f \in H_m^1(T^N)$  of (3.1) belongs to  $H^2(T^N)$  and therefore  $P_N f$  belongs to  $H^1(T^N)^N$ . We adopt the following notation.

**Definition 3.2.** For  $f \in H^1((0,1)^N)$  we define

$$Lf := \text{Id}_m^{-1}(\Delta f)$$

which has a meaning since  $\Delta f \in H^{-1}((0,1)^N)$ .

**Remark 3.3.** If in particular  $f \in H^1(T^N) \cap H^2((0,1)^N)$  satisfying Neumann boundary conditions, then  $Lf = \Delta f \in L^2((0,1)^N)^N \subset H^{-1}((0,1)^N)$  and hence in particular  $\text{Id}_m^{-1}(\Delta f) \in H_m^{-1}(T^N)$ . Hence there exists by Lemma 3.1 a unique  $u_{Lf} \in H_m^1(T^N)$  such that

$$\Delta u_{Lf} = Lf - \mu_0(Lf),$$

that is,

$$\Delta u_{Lf} = Lf,$$

Because  $f - \mu_0(f) \in H_m^1(T^N)$ , one can easily conjecture that

$$(3.4) \quad u_{Lf} = f - \mu_0(f).$$

This is indeed the case, as

$$\begin{aligned} \int_{(0,1)^N} \nabla(f - \mu_0(f)) \cdot \nabla \bar{v} \, dx &= \int_{(0,1)^N} \nabla f \cdot \nabla \bar{v} \, dx \\ &= \int_{\partial(0,1)^N} \frac{\partial f}{\partial n} \bar{v} \, dx - \int_{(0,1)^N} \Delta f \bar{v} \, dx \\ &= - \int_{(0,1)^N} \Delta f \bar{v} \, dx, \quad \forall v \in H_m^1(T^N), \end{aligned}$$

by the formula of Gauß–Green.

**Remark 3.4.** Note that our definition implies directly that

$$\mu_0(Lf) = \langle \text{Id}_m^{-1} \Delta f, 1 \rangle = 0, \quad \forall f \in H^1((0,1)^N),$$

and that  $\text{Id}_m^{-1} \Delta f = \Delta f$  in  $\mathcal{D}'((0,1)^N)$ .

**Theorem 3.5.** The space  $V$  and hence  $\tilde{V}$  are densely and compactly embedded in  $H$ .

*Proof.* Denote by  $\tilde{H}$  the closure of  $V$  in  $H$ . To prove that

$$H = \tilde{H},$$

it then suffices to show that any  $f \in H$  orthogonal to  $V$  is zero. Let  $f \in H$  be orthogonal to  $V$ , i.e.,

$$(f|g)_{H^{-1}(T^N)} = 0, \quad \forall g \in V.$$

As  $\mu_0(f) = \mu_0(g) = 0$ , we deduce that

$$\int_{(0,1)^N} \nabla u_f \cdot \nabla \bar{u}_g \, dx = 0, \quad \forall g \in V.$$

According to (3.1), we get equivalently

$$\langle f, u_g \rangle = 0, \quad \forall g \in V.$$

But we will show below that

$$(3.5) \quad \{u_g : g \in V\} + H_0 \quad \text{is dense in } H^1(T^N),$$

and therefore as  $f \in H$ , we deduce that  $f = 0$  because

$$\langle f, v \rangle = 0, \quad \forall v \in H^1(T^N).$$

It remains to prove (3.5). For  $u \in H^1(T^N)$ , we have already noticed in (2.1) that  $u - R(\gamma_0 u) \in H_0^1((0, 1)^N)$ , therefore there exists a sequence of  $\psi_n \in \mathcal{D}((0, 1)^N)$  such that

$$\psi_n \rightarrow u - R(\gamma_0 u) \text{ in } H_0^1((0, 1)^N) \text{ as } n \rightarrow \infty,$$

or equivalently

$$(3.6) \quad \psi_n + R(\gamma_0 u) \rightarrow u \text{ in } H^1(T^N) \text{ as } n \rightarrow \infty.$$

Now for  $\psi \in \mathcal{D}((0, 1)^N)$ , we notice that  $L\psi = \Delta\psi \in V$ , i.e.,  $\Delta\psi$  is orthogonal to the weakly harmonic functions: In fact if  $v$  is a weakly harmonic function, then exploiting the fact that  $\psi$  has compact support the formula of Gauß–Green yields

$$\int_{(0,1)^N} \Delta\psi \bar{v} dx = - \int_{(0,1)^N} \nabla\psi \nabla \bar{v} dx = 0,$$

where the last identity follows by definition of weakly harmonic function. Furthermore,  $u_{\Delta\psi} = \psi - \mu_0(\psi)$  by Remark 3.3.

Applying these last remarks to (3.6) means that

$$u_{\Delta\psi_n} + \mu_0(\psi_n) + R(\gamma_0 u) \rightarrow u \text{ in } H^1(T^N) \text{ as } n \rightarrow \infty.$$

As  $\mu_0(\psi_n) + R(\gamma_0 u) \in H_0$  and  $\Delta\psi_n \in V$ , the density assertion in (3.5) follows. Finally, compactness follow by the compactness of the embedding  $H^1(T^N) \hookrightarrow L^2((0, 1)^N)$ .  $\square$

We are finally in the position to prove a formula that can be seen as an  $H^{-1}$ -analogue of the usual Gauß–Green-formulae that hold with respect to the inner product of  $L^2$ .

**Lemma 3.6.** *For all  $f \in H^1(T^N)$  and all  $h \in L^2((0, 1)^N)$  one has*

$$(3.7) \quad (\text{Id}_m^{-1}(\Delta f)|h)_{H^{-1}(T^N)} = -(f|h)_{L^2(T^N)} + (R\gamma_0 f|h)_{L^2(T^N)}.$$

*Proof.* For all  $f \in H^1(T^N)$ , as  $\Delta f$  belongs to  $H^{-1}((0, 1)^N)$ , we can set  $g = \text{Id}_m^{-1}(\Delta f)$  that satisfies  $\mu_0(g) = 0$  and consequently for all  $h \in L^2((0, 1)^N)$  one has

$$(3.8) \quad (g|h)_{H^{-1}(T^N)} = \int_{(0,1)^N} \nabla u_g \cdot P_N h \, dx,$$

with

$$P_N h = \nabla u_h + \frac{\mu_0(h)}{N} \left( \vec{x} - \frac{\vec{1}}{2} \right).$$

Owing to (3.1), we get

$$\int_{(0,1)^N} \nabla u_g \cdot \nabla u_h \, dx = -\langle g, u_h \rangle,$$

and by the definition of  $\text{Id}_m^{-1}$ , we obtain

$$\int_{(0,1)^N} \nabla u_g \cdot \nabla u_h \, dx = -\langle \Delta f, u_h - R\gamma_0 u_h \rangle = -\langle \Delta(f - R\gamma_0 f), u_h - R\gamma_0 u_h \rangle.$$

Hence by the definition of  $\Delta(f - R\gamma_0 f)$  as element of  $H^{-1}((0, 1)^N)$ , we get

$$\begin{aligned} \int_{(0,1)^N} \nabla u_g \cdot \nabla u_h \, dx &= \langle \nabla(f - R\gamma_0 f) \cdot \nabla(u_h - R\gamma_0 u_h) \rangle \\ &= - \int_{(0,1)^N} (f - R\gamma_0 f) \Delta(u_h - R\gamma_0 u_h) \, dx. \end{aligned}$$

As  $R\gamma_0 u_h$  is harmonic and  $\Delta u_h = h - \mu_0(h)$ , we get

$$\begin{aligned} (3.9) \quad \int_{(0,1)^N} \nabla u_g \cdot \nabla u_h \, dx &= - \int_{(0,1)^N} (f - R\gamma_0 f)(h - \mu_0(h)) \, dx \\ &= - \int_{(0,1)^N} f h \, dx + \int_{(0,1)^N} R\gamma_0 f h \, dx + \mu_0(h) \int_{(0,1)^N} (f - R\gamma_0 f) \, dx. \end{aligned}$$

On the other hand, one sees that

$$\int_{(0,1)^N} \nabla u_g \cdot \left( \vec{x} - \frac{\vec{1}}{2} \right) dx = \int_{(0,1)^N} \nabla u_g \cdot \nabla v dx,$$

where  $v \in H^1(T^N)$  is defined by

$$v(x) := \frac{1}{2} \left\| \vec{x} - \frac{\vec{1}}{2} \right\|_{L^2}^2.$$

Hence (3.2) yields

$$\int_{(0,1)^N} \nabla u_g \cdot \left( \vec{x} - \frac{\vec{1}}{2} \right) dx = -\langle g, v \rangle,$$

and again by definition of  $\text{Id}_m^{-1}$ ,

$$\int_{(0,1)^N} \nabla u_g \cdot \left( \vec{x} - \frac{\vec{1}}{2} \right) dx = -\langle \Delta f, v - R\gamma_0 v \rangle = -\langle \Delta(f - R\gamma_0 f), v - R\gamma_0 v \rangle.$$

As before we then obtain

$$\int_{(0,1)^N} \nabla u_g \cdot \left( \vec{x} - \frac{\vec{1}}{2} \right) dx = -\langle f - R\gamma_0 f, \Delta(v - R\gamma_0 v) \rangle = -N \langle f - R\gamma_0 f, 1 \rangle.$$

This identity and (3.9) in (3.8) yields the conclusion.  $\square$

**Remark 3.7.** Let us comment on the special case  $N = 1$ . Then for all  $h \in L^2(0,1)$   $P_1 h$  is equal to  $Ph$  defined as in [9, (2.5)]. Furthermore, also in view of Remark 2.1 we find that  $V$  and  $\tilde{V}$  agree with the two spaces with same name introduced in [9, § 2]. Therefore, the theory we develop in the present note is a generalization of that introduced in [9].

#### 4. OPERATORS IN THE SPACE OF ZERO MEAN FUNCTIONS

In this section we want to determine precisely two relevant realizations of the Laplacian in  $H$ , the space of those functionals that annihilate periodic and weakly harmonic functions.

We are going to consider the sesquilinear form  $a$  defined by

$$(4.1) \quad a(f, g) := \int_{(0,1)^N} f(x) \overline{g}(x) dx,$$

with form domain either  $\tilde{V}$  or  $V$ . Since both  $\tilde{V}$  and  $V$  are dense in  $H$  by Lemma 3.5, the form  $a$  with domain  $\tilde{V}$  or  $V$  is associated with a linear operator  $(\tilde{A}, D(\tilde{A}))$  or  $(A, D(A))$ , respectively, defined by

$$\begin{aligned} D(\tilde{A}) &:= \left\{ f \in \tilde{V} : \exists g \in H : a(f, h) = \int_{(0,1)^N} \nabla u_g(x) \cdot \nabla \tilde{u}_h(x) dx \quad \forall h \in \tilde{V} \right\}, \\ \tilde{A}f &:= g \end{aligned}$$

and

$$\begin{aligned} D(A) &:= \left\{ f \in V : \exists g \in H : a(f, h) = \int_{(0,1)^N} \nabla u_g(x) \cdot \nabla \tilde{u}_h(x) dx \quad \forall h \in V \right\}, \\ Af &:= g. \end{aligned}$$

Let us describe these two operators more precisely.

**Theorem 4.1.** *One has*

$$\begin{aligned} D(\tilde{A}) &= \tilde{V} \cap (H_m^1(T^N) + V_0), \\ Af &= -\text{Id}_m^{-1} \Delta f, \quad \forall f \in D(\tilde{A}). \end{aligned}$$

Observe that in view of Remark 2.6, each function in  $D(\tilde{A})$  is weakly periodic.



*Proof.* Denote

$$\mathcal{K} := \tilde{V} \cap (H_m^1(T^N) + V_0).$$

Let us first show the inclusion  $D(\tilde{A}) \subset \mathcal{K}$ . Let  $f \in D(\tilde{A})$ . Then there exists  $g = Af \in H_m^{-1}(T^N)$  (because  $g \in H$ ) such that

$$\int_{(0,1)^N} f(x) \bar{h}(x) dx = \int_{(0,1)^N} \nabla u_g(x) \cdot \nabla \bar{u}_h(x) dx, \quad \forall h \in \tilde{V}.$$

But according to (3.1) we then have equivalently

$$\int_{(0,1)^N} f(x) \bar{h}(x) dx = -\overline{\langle h, u_g \rangle} = -\int_{(0,1)^N} \bar{h}(x) u_g(x) dx, \quad \forall h \in \tilde{V},$$

because  $h$  belongs to  $L^2((0,1)^N)$ . This means equivalently that  $f + u_g$  is orthogonal (in the  $L^2$  sense) to  $\tilde{V}$ , i.e.,  $f + u_g$  belongs to  $V_0$ . As elements of  $V_0$  are harmonic functions, we deduce that

$$\Delta(f + u_g) = 0 \text{ in } \mathcal{D}'((0,1)^N),$$

and by (3.3), we find

$$\Delta f = -g \text{ in } \mathcal{D}'((0,1)^N).$$

This shows that  $\Delta f$  belongs to  $H^{-1}((0,1)^N)$  and reminding that  $g$  belongs to  $H$ , we get further

$$g = -\text{Id}_m^{-1}(\Delta f).$$

Similarly as element of  $V_0$  are weakly periodic and  $u_g \in H^1(T^N)$  (hence weakly periodic), we deduce that  $f$  is weakly periodic.

Let us now prove the converse inclusion. Let  $f \in \mathcal{K}$  then  $g = -\text{Id}_m^{-1}(\Delta f)$  belongs to  $H$  and by Lemma 3.6, we get for any  $h \in \tilde{V}$

$$(g|h)_H = (f|h)_{L^2},$$

as  $(R\gamma_0 f|h)_{L^2(T^N)} = 0$ . This shows that  $f \in D(\tilde{A})$  and concludes the proof.  $\square$

**Theorem 4.2.** *One has*

$$\begin{aligned} D(A) &= V \cap (H_m^1(T^N) + V_1), \\ Af &= -\text{Id}_m^{-1} \Delta f, \quad \forall f \in D(A). \end{aligned}$$

*Proof.* Denote

$$\mathcal{K} := V \cap (H_m^1(T^N) + V_1).$$

Let us first show the inclusion  $D(A_0) \subset \mathcal{K}$ . Let  $f \in D(A_0)$ . Then there exists  $g =: A_0 f \in H \equiv H_m^{-1}(T^N)$  such that

$$\int_{(0,1)^N} f(x) \bar{h}(x) dx = \int_{(0,1)^N} \nabla u_g(x) \cdot \nabla \bar{u}_h(x) dx, \quad \forall h \in V.$$

But according to (3.1) we then have equivalently

$$\int_{(0,1)^N} f(x) \bar{h}(x) dx = -\overline{\langle h, u_g \rangle} = -\int_{(0,1)^N} \bar{h}(x) u_g(x) dx, \quad \forall h \in V,$$

because  $h$  belongs to  $L^2((0,1)^N)$ . This means equivalently that  $f + u_g$  is orthogonal (in the  $L^2$  sense) to  $V$ , i.e.,  $f + u_g$  belongs to  $V_1$ . As elements of  $V_1$  are harmonic functions, we deduce that

$$\Delta(f + u_g) = 0 \text{ in } \mathcal{D}'((0,1)^N),$$

and by (3.3), we find

$$\Delta f = -g \text{ in } \mathcal{D}'((0,1)^N).$$

This shows that  $\Delta f$  belongs to  $H^{-1}((0,1)^N)$  and reminding that  $g$  belongs to  $H$ , we get further

$$g = -\text{Id}_m^{-1}(\Delta f).$$

This shows the desired inclusion because by definition  $D(A_0) \subset V$ .

The converse inclusion is proved as in the previous Theorem by using Lemma 3.6.  $\square$

**Remark 4.3.** If one wants to study the Laplacian acting on functions on a more general domain  $\Omega \subset \mathbb{R}^N$ , one may look for a Lipschitz-homeomorphism mapping  $(0, 1)^N$  into  $\Omega$ . In this way, however, also the Laplacian on  $(0, 1)^N$  is transformed into a different elliptic operator on  $\Omega$ . The following is a possibly more convenient approach: Take a “partition”  $\Gamma = \{\Gamma_1, \dots, \Gamma_{2N}\}$  of  $\partial\Omega$  such that

- each set  $\Gamma_i \subset \partial\Omega$  is open,
- the sets  $\Gamma_i$  are pairwise disjoint,
- the union of their closures covers  $\partial\Omega$ , and
- such that each  $\Gamma_i$  is bijective (say, via some  $\theta_i$ ) to  $\Gamma_{N+i}$  for all  $i = 1, \dots, N$ .

Define

$$H_\Gamma^1(\Omega) := \{u \in H^1((\Omega)) : \gamma_i u = \gamma_{N+i} u \circ \theta_i \text{ on } \Gamma_i, \forall i = 1, \dots, N\},$$

where  $\gamma_i$  is the trace operator on  $\Gamma_i$ . In this way, we can regard in a natural way the space  $H^1(T^N)$  as the space  $H_\Gamma^1((0, 1)^N)$  with the partition  $\Gamma$  made of faces of the hypercube, and deduce that all the constructions of this paper can be extended to consider realizations (with linear conditions on the moments of order 0 and 1) of the Laplacians acting on functions over general domains  $\Omega$ .

## 5. WELL-POSEDNESS OF THE PARABOLIC PROBLEM IN THE SPACE OF ZERO MEAN FUNCTIONS

In the previous section we have showed that the densely defined sesquilinear, symmetric, forms  $a$  with domain  $V$  is bounded and coercive. Hence, from the general theory of forms we deduce the following.

**Proposition 5.1.** *Both operators  $A$  and  $\tilde{A}$  generate analytic, contractive, exponentially stable and compact semigroups on  $H$ .*

Hence, we can say that the abstract Cauchy problems associated with  $A$  and  $\tilde{A}$  are well-posed. However, the challenge is now to understand just *which* are these abstract Cauchy problem. The characterization of  $D(A)$  and  $D(\tilde{A})$  in Theorem 4.1 and Theorem 4.2 is not very satisfying, and hence neither is the description of the differential equations effectively solved by the semigroups. Yet, we are able to deduce well-posedness of the classical heat equation in the following weak sense.

**Theorem 5.2.** *Let  $u_0 \in D(A)$ . Then the function*

$$t \mapsto u(t, \cdot) := e^{-tA} u_0(\cdot)$$

*solves*

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x), \quad t > 0,$$

*with conditions*

$$(5.1) \quad \int_{(0,1)^N} u(t, y) \bar{h}(y) dy = 0 \quad \forall h \in H_1, \quad t > 0,$$

*in the sense of distributions, i.e., for all  $t > 0$  (5.1) is satisfied and moreover*

$$\left\langle \frac{\partial u}{\partial t}(t, \cdot) - \Delta u(t, \cdot), g \right\rangle_{H-H'} = 0 \quad \text{for all } g \in H_0^1((0, 1)^N).$$

*Proof.* It suffices to take into account Theorem 4.2 and recall that the analytic semigroup  $(e^{-tA})_{t \geq 0}$  maps  $u_0$  into the domain of  $D(A)$  for all  $t > 0$ , and moreover  $u$  solves

$$\left\langle \frac{\partial u}{\partial t}(t, \cdot) - \text{Id}_m^{-1} \Delta u(t, \cdot), g \right\rangle_{H-H'} = 0 \quad \text{for all } g \in H',$$

Now,  $H_0$  is a closed subspace of  $H^1(T^N)$  and  $H$  is its annihilator, we can deduce that there is an isomorphism

$$\mathfrak{J} : H^1(T^N) / H_0 \rightarrow H',$$

given by

$$\langle \varphi, \mathfrak{J}[h] \rangle_{H-H'} := \langle \varphi, h \rangle_{H^{-1}(T^N) - H^1(T^N)}, \forall \varphi \in H.$$

Then we have that

$$\left\langle \frac{\partial u}{\partial t}(t, \cdot) - \text{Id}_m^{-1} \Delta u(t, \cdot), g \right\rangle_{H-H'} = 0 \quad \text{for all } g \in H^1(T^N),$$

and in particular

$$\left\langle \frac{\partial u}{\partial t}(t, \cdot) - \text{Id}_m^{-1} \Delta u(t, \cdot), g \right\rangle_{H-H'} = 0 \quad \text{for all } g \in \mathcal{D}((0, 1)^N).$$

Now, the assertion follows from Remark 3.4. □

Similarly, exploiting instead Theorem 4.1, we can obtain the following.

**Theorem 5.3.** *Let  $u_0 \in D(\tilde{A})$ . Then the function*

$$t \mapsto u(t, \cdot) := e^{-t\tilde{A}}u_0(\cdot)$$

*solves*

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x), \quad t > 0,$$

*in the sense of distributions, it is periodic in the sense of Definition 2.4 and satisfies conditions*

$$(5.2) \quad \int_{(0,1)^N} u(t, y) \bar{h}(y) dy = 0 \quad \forall h \in H_0, \quad t > 0.$$

Observe that in particular the constant functions belong to  $H_0$ , and therefore  $H$  contains all mean zero functions – i.e., all functions with vanishing moment of order 0. Thus, if  $f \in \tilde{V}$  (and in particular if  $f \in D(\tilde{A})$ ), then

$$\int_{(0,1)^N} f(x) dx = 0.$$

Furthermore, the functions  $g_i : (0, 1)^N \ni x = (x_1, \dots, x_n) \mapsto x_i \in \mathbb{R}$  belong to  $H_1$  for all  $i = 1, \dots, n$ . Hence: If  $f \in V$  (and in particular if  $f \in D(A)$ ), then

$$\int_{(0,1)^N} f(x) dx = \int_{(0,1)^N} x_i f(x) dx = 0, \quad i = 1, \dots, N.$$

In particular, for all  $t > 0$  both  $(e^{tA})_{t \geq 0}$  and  $(e^{t\tilde{A}})_{t \geq 0}$  map any  $u_0 \in H$  into a function that has mean zero, and furthermore  $(e^{tA})_{t \geq 0}$  maps any  $u_0 \in H$  into a function that has vanishing *first linear moments along each axis*, as they are defined in [11, § 9.6.5].

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